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THE THEORY OF HIGHER ORDER WEIGHT FUNCTIONS FOR LINEAR ELASTIC PLANE PROBLEMS

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Abstract—We generalize Bueckner's fundamental field concept and develop higher order weight functions for calculating power expansion coefficients of a regular elastic field in a two-dimensional body in the absence of body forces. Problems of the first and third kind are investigated. Integral formulas for the expansion coefficients are given for interior points and crack tips. In these formulas the integration is performed over the boundary of the body, crack faces included. The prescribed boundary data (tractions and/or displacements) of the regular field appear in the integrand in weighted form. The weights are derived from fundamental fields of universal character. The significance of these expansion coefficients in fracture analysis is also discussed.

1. INTRODUCTION

The weight function theory was introduced by Bueckner (1970, 1973) for determining stress intensity factors in a linear elastic cracked body. The weight functions are universal functions for the given crack geometry and the stress intensity factors under any applied loading can be calculated by using the weight functions through quadrature. Bueckner's theory is based on the concept of a fundamental field (see subsequent discussion) and Betti's theorem of reciprocity. A different interpretation of Bueckner's weight functions was given by Rice (1972) through the notion of energetic forces and crack front motion.

Subsequent studies of the weight functions were carried out by Bueckner (1975), Paris *et al.* (1976), Labbens *et al.* (1976b), Wu and Carlsson (1983), Bortman and Banks-Sills (1983), Kirchner (1986), and Kirchner and Michot (1986), among others. Recent advances of the theory in three-dimensions have been given by Rice (1985a, 1985b) and Bueckner (1977, 1987), and some applications of the three-dimensional theory can be found in the works of Labbens *et al.* (1976a), Sham and Zhou (1989), and Gao and Rice (1986, 1987).

In a regular field of plane deformation without body forces, the complex stress intensity factor at the tip of a traction free crack determines the most significant expansion coefficients of Muskhelishvili's analytic field functions. Since the work of Irwin (1957), there has been a growing interest in the higher order expansion coefficients associated with cracks and their stability. The next term in the power series expansion also plays an important role in fracture analyses. This term corresponds to a uniform normal stress acting parallel to the faces of a Mode I traction free crack and it is often referred to as the elastic T-term. Larsson and Carlsson (1973) and Rice (1974) have shown that the inclusion of the elastic T-term in the small-scale yielding procedure of elastic–plastic fracture can increase the range of load levels over which such a procedure gives accurate results. Recent work by Bilby *et al.* (1986) has also indicated that the inclusion of T extends the range of validity of the small-scale yielding conditions at finite strains. As demonstrated by Cotterell and Rice (1980), another significance of T in fracture analysis is that it governs the stability of a straight crack path under Mode I loading conditions. Because of these features, the elastic T-term serves as a biaxial parameter and it is often used together with the stress intensity factor to characterize fracture (Larsson and Carlsson, 1973; Rice, 1974; Leevers and Radon, 1982).

The succeeding higher order coefficients in the power series expansion are also of great importance in certain experimental techniques for measuring stress intensity factors. These techniques include: photoelasticity (Theocaris and Gdoutos, 1975; Etheridge and Dally, 1978; Sanford *et al.*, 1981; Chona *et al.*, 1983; Barker *et al.*, 1985); strain-gage method

(Dally and Sanford, 1987); reflected caustics in optically isotropic materials (Theocaris and Ioakimidis, 1979); and optically anisotropic materials (Phillips and Sanford, 1981). The inclusion of the higher order coefficients permits a more accurate interpretation of fracture data obtained at finite distances from the crack tip. For example, Etheridge and Dally (1978) used two additional coefficients, Phillips and Sanford (1981) used four, and Dally and Sanford (1987) used three, in the analysis of the experimental data.

The expansion coefficients for interior points are also of great importance in stress analysis. For example, the coefficient of the term linear in z can be related to the force on a discrete screw dislocation in an elastic body (Sham, 1988c).

In this work we are concerned with the calculation of such coefficients at interior points as well as at crack tips. To this end Bueckner's (1970) fundamental fields are generalized. As in his theory, the reciprocity theorem is applied to the regular field, the expansion coefficients of which we wish to determine, and to an appropriate fundamental field. This procedure leads to integral representations of the coefficients. The integral extends over the boundary of the elastic body; the prescribed boundary data (tractions and/or displacements) of the regular field appear in the integrand, multiplied in work-like manner by the energy-conjugate data of the fundamental field as weight function.

The synopsis of this paper is as follows. Preliminaries are introduced in Section 2, including some general results to be used subsequently. In Section 3, Bueckner's (1970, 1973) fundamental field concept and elastic reciprocity are used to develop integration formulas for determining these expansion coefficients at interior points and crack tips through the higher order weight functions. Modifications to the integration formulas for problems involving both traction and displacement boundaries are given in Section 4. The construction of fundamental fields for closed-crack geometry is discussed in Section 5. In Section 6, generalization of the results to infinite domains is given. Integration formulas are also applied to evaluate the expansion coefficients of a semi-infinite crack in an infinite body under certain special loading conditions. These specific applications are chosen because results for the fundamental fields and the stress-analysis problems can be obtained in closed form, and thus allowing a demonstration of the soundness of the theory. Further extension of the theory to include body forces in the elastic field is also discussed in Section 6.

Recently, Sham and Bueckner (1988) have employed the concepts of the fundamental field and elastic reciprocity to develop weight functions for determining the notch-interface stress intensity factor in a piecewise homogeneous, isotropic body deforming in antiplane strain. The theory introduced in this paper can be generalized to determine higher order expansion coefficients for notch tips. By using Rice's energetic approach, Parks (1979) has given a procedure to calculate the stress concentration factor by means of weight functions. It can be noted that the present theory can also be extended to determine expansion coefficients at smooth boundary locations.

2. PRELIMINARIES

Consider an elastic body under plane deformations. Let x, y be the Cartesian coordinates and $z \equiv x+iy$ be a complex variable with i being the imaginary unit. Then in the absence of body forces any elastic field in the body may be expressed in terms of two analytic functions of z, $\phi(z)$ and $\psi(z)$ (Muskhelishvili, 1977). However, it is more convenient to express the elastic state by the analytic functions ϕ and ρ, where $\rho \equiv \psi + z\phi'$. The displacements, u and v, and the in-plane stresses are then given by

$$2\mu w \equiv 2\mu(u+iv) = \kappa\phi(z) - \overline{\rho(z)} + (\bar{z}-z)\overline{\phi'(z)} \tag{1a}$$

$$\sigma_{xx} = \mathrm{Re}\,[\phi'(z) + 2\phi'(z) - \overline{\rho'(z)} + (\bar{z}-z)\overline{\phi''(z)}] \tag{1b}$$

$$\sigma_{yy} = \mathrm{Re}\,[\phi'(z) + \overline{\rho'(z)} - (\bar{z}-z)\overline{\phi''(z)}] \tag{1c}$$

$$\sigma_{xy} = -\mathrm{Im}\,[\phi'(z) - \rho'(z) + (z-\bar{z})\phi''(z)] \tag{1d}$$

with

$$\kappa = 3-4\nu \quad \text{(plane strain)}$$

$$\kappa = \frac{3-\nu}{1+\nu} \quad \text{(generalized plane stress)}$$

where μ is the shear modulus and ν Poisson's ratio.

The preceding representation is not without ambiguity. Necessary and sufficient to yield a field of vanishing stresses are functions $\phi(z)$, $\rho(z)$ of the form

$$\phi(z) = a+icz, \quad \rho(z) = b+icz \tag{1e}$$

with arbitrary complex coefficients a, b and an arbitrary real coefficient c. The associated displacement is a rigid body motion given by

$$2\mu w = \kappa a-\bar{b}+i(\kappa+1)cz. \tag{1f}$$

The components of the traction vector attacking the material to the left of an oriented arc element ds in the x- and y-directions are X and Y respectively and they are given by

$$Z\,ds \equiv (X+iY)\,ds = -i\,dP \tag{2a}$$

where

$$P = P(z) = \phi(z)+\overline{\rho(z)}+(z-\bar{z})\overline{\phi'(z)}; \tag{2b}$$

$P(z)$ is not an analytic function in general. Also, if $N \equiv n_x+in_y$ where n_x and n_y are the Cartesian components of the unit normal pointing to the right of the oriented arc element ds

$$\frac{dz}{ds} = iN, \quad \frac{d\bar{z}}{ds} = -i\bar{N}$$

and we may express the tractions as

$$Z = N[\phi'+\overline{\phi'}]+\bar{N}[\overline{\phi'}-\overline{\rho'}+(\bar{z}-z)\overline{\phi''}]. \tag{2c}$$

Using eqn (2b), the displacments may be written in the form

$$2\mu w = (1+\kappa)\phi-P. \tag{3}$$

The resultant force and the resultant moment about the origin, produced by the tractions acting on an oriented curve Π, are respectively

$$-i\int_{\Pi} dP \quad \text{and} \quad -\mathrm{Re}\int_{\Pi} \bar{z}\,dP. \tag{4}$$

Let ϕ, ρ be regular along Π; this makes P continuous on Π. Let Π be either closed or, if open, such that P takes the same value at its end points. In this case integral $(4)_1$ vanishes, i.e. the tractions acting on Π have zero force resultant. As for the moment, integration by parts is used and it is found quite generally

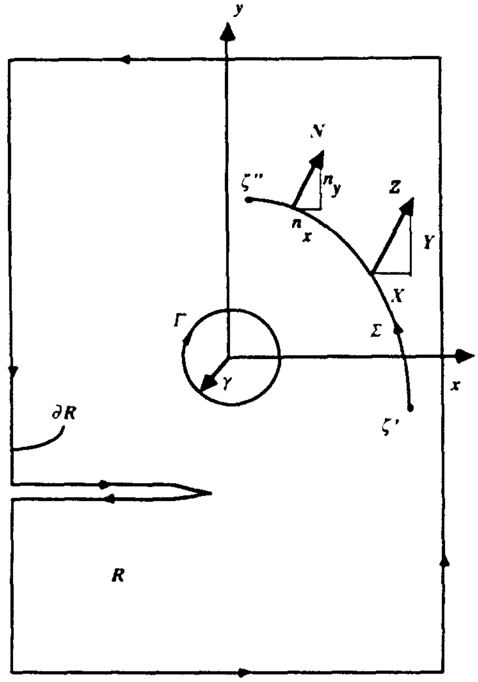

Fig. 1. Finite plane elastic body R with bounding surface ∂R. Γ is a small circle centered at an interior point, $z = 0$, and $\Pi \equiv \partial R \cup \Gamma$ bounds a sub-domain of R.

$$
\begin{aligned}
M &= -\mathrm{Re}\int_\Pi \bar{z}\,\mathrm{d}P = \mathrm{Re}\int_\Pi P\,\mathrm{d}\bar{z} - \mathrm{Re}\,P\bar{z}\Big]_{\zeta'}^{\zeta''} = \mathrm{Re}\int_\Pi \bar{P}\,\mathrm{d}z - \mathrm{Re}\,P\bar{z}\Big]_{\zeta'}^{\zeta''} \\
&= \mathrm{Re}\int_\Pi [\bar{\phi}+\rho+(\bar{z}-z)\phi']\,\mathrm{d}z - \mathrm{Re}\,P\bar{z}\Big]_{\zeta'}^{\zeta''} \\
&= \mathrm{Re}\int_\Pi [\bar{\phi}\,\mathrm{d}z - \phi\,\mathrm{d}\bar{z} + (\phi+\rho)\,\mathrm{d}z] + \mathrm{Re}\,[(\bar{z}-z)\phi - P\bar{z}]\Big]_{\zeta'}^{\zeta''} \\
&= \mathrm{Re}\int_\Pi [\phi+\rho]\,\mathrm{d}z + \mathrm{Re}\,[(\bar{z}-z)\phi - P\bar{z}]\Big]_{\zeta'}^{\zeta''}
\end{aligned}
\tag{5}
$$

where another integration by parts was used to convert the integral with ϕ'; and ζ', ζ'' are the end points of Π.

Consider two elastic states distinguished by the subscripts 1, 2 for the plane body R shown in Fig. 1. Let Σ be an oriented, piecewise smooth curve in R, going from a point ζ' to another point ζ''. Along Σ, the work done by the tractions of state j on the displacements of state k is given by

$$
\begin{aligned}
W_{jk}(\Sigma) &= \int_\Sigma (X_j u_k + Y_j v_k)\,\mathrm{d}s = \mathrm{Re}\int_\Sigma (X+\mathrm{i}Y)_j(u-\mathrm{i}v)_k\,\mathrm{d}s \\
&= -\mathrm{Re}\,\mathrm{i}\int_\Sigma \bar{w}_k\,\mathrm{d}P_j = \mathrm{Im}\int_\Sigma \bar{w}_k\,\mathrm{d}P_j; \quad j,k = 1,2.
\end{aligned}
\tag{6}
$$

Consider the difference in works

$$
W^*(\Sigma) \equiv W_{12} - W_{21} = \mathrm{Im}\int_\Sigma (\bar{w}_2\,\mathrm{d}P_1 - \bar{w}_1\,\mathrm{d}P_2) \tag{7a}
$$

which, by (3), may be written as

$$W^*(\Sigma) = \frac{1}{2\mu} \operatorname{Im} \int_\Sigma [\bar{P}_1 \, dP_2 - \bar{P}_2 \, dP_1 + (1+\kappa)(\bar{\phi}_2 \, dP_1 - \bar{\phi}_1 \, dP_2)]. \tag{7b}$$

Obviously

$$\operatorname{Im} \int_\Sigma (\bar{P}_1 \, dP_2 - \bar{P}_2 \, dP_1) = \operatorname{Im} \int_\Sigma (\bar{P}_1 \, dP_2 + P_2 \, d\bar{P}_1) = \operatorname{Im} \bar{P}_1 P_2 \Big]_{\zeta'}^{\zeta''}.$$

Furthermore

$$\operatorname{Im} \int_\Sigma (\bar{\phi}_2 \, dP_1 - \bar{\phi}_1 \, dP_2) = \operatorname{Im} \int_\Sigma (P_2 \, d\bar{\phi}_1 - P_1 \, d\bar{\phi}_2) + \operatorname{Im} [\bar{\phi}_2 P_1 - \bar{\phi}_1 P_2]\Big]_{\zeta'}^{\zeta''}$$

by integration by parts. More explicitly

$$\operatorname{Im} \int_\Sigma [P_2 \, d\bar{\phi}_1 - P_1 \, d\bar{\phi}_2] = \operatorname{Im} \int_\Sigma [\bar{P}_1 \, d\phi_2 - \bar{P}_2 \, d\phi_1]$$

$$= \operatorname{Im} \int_\Sigma [\bar{\phi}_1 \, d\phi_2 - \bar{\phi}_2 \, d\phi_1 + \rho_1 \, d\phi_2 - \rho_2 \, d\phi_1 + (\bar{z} - z)(\phi'_1 \, d\phi_2 - \phi'_2 \, d\phi_1)].$$

Here

$$\phi'_1 \, d\phi_2 - \phi'_2 \, d\phi_1 \equiv 0;$$

also

$$\operatorname{Im} [\bar{\phi}_1 \, d\phi_2 - \bar{\phi}_2 \, d\phi_1] = \operatorname{Im} [\bar{\phi}_1 \, d\phi_2 + \phi_2 \, d\bar{\phi}_1] = \operatorname{Im} d[\bar{\phi}_1 \phi_2]$$

hence

$$\operatorname{Im} \int_\Sigma [P_2 \, d\bar{\phi}_1 - P_1 \, d\bar{\phi}_2] = \operatorname{Im} \int_\Sigma [\rho_1 \, d\phi_2 - \rho_2 \, d\phi_1] + \operatorname{Im} \bar{\phi}_1 \phi_2 \Big]_{\zeta'}^{\zeta''}.$$

Putting all pieces together one obtains

$$W^*(\Sigma) = \frac{1+\kappa}{2\mu} \operatorname{Im} \int_\Sigma (\rho_1 \, d\phi_2 - \rho_2 \, d\phi_1) + \frac{1}{2\mu} \operatorname{Im} [\bar{P}_1 P_2 + (1+\kappa)(\bar{\phi}_2 P_1 - \bar{\phi}_1 P_2 + \bar{\phi}_1 \phi_2)]\Big]_{\zeta'}^{\zeta''}.$$

An equivalent representation is obtained from

$$\int_\Sigma \rho_1 \, d\phi_2 = -\int_\Sigma \phi_2 \, d\rho_1 + \rho_1 \phi_2 \Big]_{\zeta'}^{\zeta''}.$$

This leads to

$$W^*(\Sigma) = -\frac{1+\kappa}{2\mu}\,\mathrm{Im}\left[\int_\Sigma (\rho_2\,d\phi_1+\phi_2\,d\rho_1)+\mathrm{Im}\,G\right]_{z'}^{z''} \tag{8}$$

with

$$\begin{aligned} G &= \frac{1}{2\mu}[[\bar{P}_1-(1+\kappa)\bar{\phi}_1]P_2+(1+\kappa)[P_1-(\phi_1+\bar{\rho}_1)]\bar{\phi}_2] \\ &= -\bar{w}_1P_2+\frac{1+\kappa}{2\mu}[P_1-(\phi_1+\bar{\rho}_1)]\bar{\phi}_2 \\ &= -\bar{w}_1P_2+\frac{1+\kappa}{2\mu}(z-\bar{z})\overline{\phi_1'}\,\overline{\phi_2}. \end{aligned} \tag{9}$$

The preceding formulas can be extended towards a union of oriented arcs.

Before we proceed to develop the weight function theory, we first distinguish two kinds of elastic fields; namely, *regular* fields and *singular* fields. Regular fields are fields of displacements, strains and stresses which produce finite elastic energy of deformation in any sub-domain of the body. Continuously imposed displacements and/or tractions along the boundary yield regular fields in particular. We permit the strains and stresses of a regular field to be unbounded at certain boundary points, and an example is the elastic K-field for cracked solids. Singular fields are elastic fields which generate infinite elastic energy of deformation in the neighborhood of a special (singular) point. Examples of these fields are solutions to elastic boundary value problems of point forces and dipoles and fundamental fields in notched and cracked bodies (Bueckner, 1970; Rice, 1972; Sham and Bueckner, 1988).

3. PROBLEMS OF THE FIRST KIND

3.1. *Interior points*

Consider a body R of finite size loaded by prescribed tractions on ∂R where ∂R is the boundary of R (Fig. 1). It is assumed that there are no body forces. The elastic field in R is assumed to be regular. In the neighborhood of any interior point, say the origin, $z = 0$ in R, the field functions ϕ, ρ have expansions

$$\phi(z) = \sum_{n=0}^{\infty} a_n z^n,\quad \rho(z) = \sum_{n=0}^{\infty} b_n z^n,\quad a_n, b_n = \text{complex coefficients}. \tag{10}$$

Here it can be assumed $a_0 = 0$, $b_0 = 0$. This is no essential loss of generality since it effects rigid body motion only.

Next, consider a singular field

$$\phi_m^s(z) = \frac{2\mu}{1+\kappa}A_m z^{-m},\quad \rho_m^s(z) = \frac{2\mu}{1+\kappa}B_m z^{-m} \tag{11}$$

where m is a positive integer and A_m and B_m are some complex coefficients. Since ϕ_m^s and ρ_m^s and their derivatives are continuous everywhere in R except at $z = 0$, they yield a continuous P and thus a zero force resultant on any piecewise smooth contour in R which does not pass the origin. Due to (5) the corresponding resultant moment is also zero exept when the contour encircles the origin and $m = 1$. For $m = 1$, the resultant moment is $[(-4\pi\mu)/(1+\kappa)]\,\mathrm{Im}\,(A_1+B_1)$ if the contour is traversed in an anti-clockwise direction around $z = 0$. The elastic field (11) gives rise to surface tractions on ∂R. These surface tractions will be relieved by a complementary regular field, ϕ_m^r and ρ_m^r, to obtain

$$\phi_m^f = \phi_m^s + \phi_m^r, \quad \rho_m^f = \rho_m^s + \rho_m^r \tag{12}$$

where the elastic state characterized by ϕ_m^f and ρ_m^f will be referred to as a *fundamental field* of order m. The fundamental field so constructed has no body forces and it induces zero tractions on ∂R. It is clear that the regular field ϕ_m^r, ρ_m^r does not exist for $m = 1$ in general. In this case we restrict (11) by

$$\text{Im } (A_1 + B_1) = 0 \tag{13}$$

in order to ascertain (12).

The two elastic states considered in the previous section are chosen as follows. The elastic state ϕ_1 and ρ_1 is the regular field with expansions (10) near $z = 0$. The second elastic state is the fundamental field; namely $\phi_2 = \phi_m^f$ and $\rho_2 = \rho_m^f$. Let Π be a boundary which consists of ∂R and Γ (Fig. 1). Here Γ is a small circle of radius γ centered at the origin. Π bounds a sub-domain of R in which the two states are regular. The reciprocity theorem applies and one may write

$$W^*(\Pi) = W^*(\partial R) + W^*(\Gamma) = 0. \tag{14}$$

Since the first term of eqn (14) is independent of γ so must be the second term $W^*(\Gamma)$. Now $\mathrm{d}P_2$ vanishes on ∂R; thus, using (7a), one obtains for the first term

$$W^*(\partial R) = \text{Im} \int_{\partial R} \bar{w}_2 \, \mathrm{d}P_1. \tag{15}$$

Near the origin, $\phi_2 \approx \phi_m^s$ and $\rho_2 \approx \rho_m^s$. With the aid of (8), one can express $W^*(\Gamma)$ as

$$W^*(\Gamma) = -\frac{1+\kappa}{2\mu} \text{Im} \int_\Gamma (\rho_m^s \, \mathrm{d}\phi_1 + \phi_m^s \, \mathrm{d}\rho_1) + O(\gamma). \tag{16}$$

The boundary term of (8) does not contribute because Γ is a closed path and function G is continuous on Γ. The term $O(\gamma)$ stands for and represents the order of the contribution of ϕ_m^r, ρ_m^r. (Actually that contribution vanishes.) One now substitutes eqns (10) and (11) into eqn (16) to obtain

$$\begin{aligned} W^*(\Gamma) &= -\text{Im} \sum_{n=1}^{\infty} \int_\Gamma (B_m a_n + A_m b_n) n z^{n-m-1} \, \mathrm{d}z + O(\gamma) \\ &= 2\pi m \text{ Re } [B_m a_m + A_m b_m] + O(\gamma). \end{aligned} \tag{17}$$

In the limit $\gamma \to 0$ one obtains the following formula for determining the coefficients a_m and b_m:

$$\text{Re } (B_m a_m + A_m b_m) = -\frac{1}{2\pi m} \text{Im} \int_{\partial R} \bar{w}_2 \, \mathrm{d}P_1. \tag{18}$$

Generally, four fundamental fields are needed in order to determine the two complex coefficients a_m and b_m. These fundamental fields are obtained by choosing the coefficients

Table 1. The values of A_m, B_m to be set in (11) for obtaining the appropriate fundamental field which determines the coefficients of expansions, a_m, b_m, of the elastic field at interior points

Coefficient to be determined	A_m	B_m
Re a_m	0	1
Im a_m	0	$-$i
Re b_m	1	0
Im b_m	$-$i	0

A_m and B_m in eqns (11) and (12) as in Table 1. It should be kept in mind that the complementary regular fields ϕ_m^r, ρ_m^r depend on the choice of A_m and B_m.

For $m = 1$, A_1 and B_1 are not arbitrary but are restricted by (13). One can determine Re a_1 and Re b_1 by choosing the pair (A_1, B_1) to be respectively (0, 1) and (1, 0) as before. But one can only determine the combination Im $(a_1 - b_1)$, by setting $(A_1, B_1) = (\mathrm{i}, -\mathrm{i})$. However, for a pure traction boundary value problem, more information on a_1, b_1 is of no interest since its effect shows up in a rigid body rotation only. If we denote the displacements of the fundamental fields of order m by u_m^f, v_m^f, then (18) may be rewritten as

$$\mathrm{Re}\,[B_m a_m + A_m b_m] = -\frac{1}{2\pi m}\int_{\partial R}[X_1 u_m^f + Y_1 v_m^f]\,\mathrm{d}s. \tag{19}$$

In analogy to the weight functions introduced by Bueckner (1970, 1973) for determining stress intensity factors in cracked bodies, the displacements u_m^f and v_m^f are referred to as weight functions of order m for the interior point $z = 0$. It is noted that for problems of the first kind, the weight functions are unique up to an arbitrary rigid body motion.

3.2. *Crack tips*

3.2.1. *Regular fields and fundamental fields.* We now turn to the consideration of expansion coefficients at crack tips. Consider an open crack in a finite body loaded by continuously imposed tractions on the external boundary ∂R (Fig. 2). We shall employ

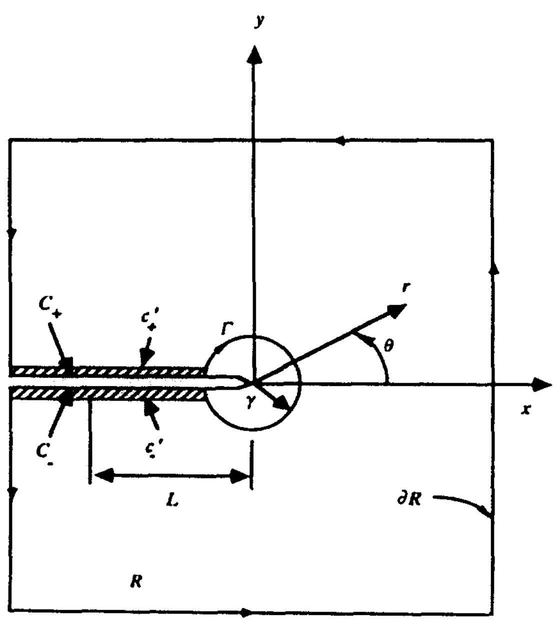

Fig. 2. An open crack in a finite body R with a bounding surface consisting of ∂R and C_+, C_- (upper and lower crack faces). Crack face loadings of the induced type are admitted in the interval $(-L, 0)$. C'_+ and C'_- are the portions of C_+ and C_- outside Γ. $\Omega \equiv \partial R \cup C'_+ \cup \Gamma \cup C'_-$ bounds a sub-domain of R.

both Cartesian (x, y) and polar (r, θ) coordinates with the origins being placed at the crack tip. It is assumed that there are no body forces present in the body. We admit loadings on some portions of the crack faces, C_+ and C_-. However, we restrict the crack face loadings to be of the induced type and this means that the traction vectors at two opposing points on the crack faces are equal and opposite to one another. Near the crack tip, we consider a continuous crack face loading in the interval $(-L, 0]$, and we shall represent this crack face loading by a convergent series for the said interval as

$$\sigma_{yy}+i\sigma_{xy} = f(x) = \sum_{k=0}^{\infty} f_k x^k, \quad -L < x \leqslant 0, \quad \text{on } C_+, C_- \tag{20}$$

where f_k is complex in general. A theorem given by Bueckner (1973) states that if the elastic field has finite energy in the neighborhood of the crack tip, and if ϕ' and ρ' are continuous in the neighborhood of the crack tip (excluding $z = 0$), then the functions ϕ and ρ admit expansions

$$\phi(z) = \sum_{n=1}^{\infty} a_n z^{n/2}, \quad \rho(z) = \sum_{n=1}^{\infty} b_n z^{n/2} \tag{21a}$$

for $|z| < L$. In particular

$$b_n = \bar{a}_n \text{ for } n = \text{odd}, \quad \frac{n}{2} b_n = f_{(n-2)/2} - \frac{n}{2} \bar{a}_n \text{ for } n = \text{even}. \tag{21b}$$

It has been assumed that the origin remains fixed.

Consider the following singular field for the cracked body:

$$\phi_m^{s\prime} = \frac{2\mu}{1+\kappa} A_m z^{-m/2}, \quad \rho_m^{s\prime} = (-1)^{m+1} \frac{2\mu}{1+\kappa} \bar{A}_m z^{-m/2}, \tag{22}$$

where m = positive integer, A_m = complex coefficient. For this field $P = 0$ on the negative x-axis; it therefore gives zero tractions on the crack faces but the tractions are non-zero on any oriented, piecewise smooth curve, Γ^*, originating from a point ζ' on the lower crack face to any point ζ'' on the upper crack face. However, these tractions lead to a zero resultant force on Γ^* and this follows directly from $(4)_1$. The resultant moment, M, produced by these tractions is, from (5),

$$M = \mathrm{Re} \int_{\Gamma^*} [\rho_m^{s\prime} + \phi_m^{s\prime}] \, dz.$$

Now let $\zeta'' = \zeta'$. We find that the integral is zero for even m, $m \neq 2$ and it is purely imaginary for odd m. Thus M is zero for all singular fields $\phi_m^{s\prime}$ and $\rho_m^{s\prime}$ except $m = 2$. For $m = 2$, we obtain $M = [(-8\pi\mu)/(1+\kappa)] \, \mathrm{Im}\, A_2$.

The tractions of the singular field (22) on the external boundary ∂R are non-zero in general. Since we can choose Γ^* to be ∂R, the tractions on ∂R are self-equilibrated for $m \neq 2$. We shall remove these surface tractions in the customary manner by a complementary regular field, $\phi_m^{r\prime}$ and $\rho_m^{r\prime}$. This regular field can be represented by

$$\phi_m^{r\prime} = \sum_{j=1}^{\infty} B_j z^{j/2}, \quad \rho_m^{r\prime} = \sum_{j=1}^{\infty} (-1)^{j+1} \bar{B}_j z^{j/2}; \quad B_j = \text{complex coefficient} \tag{23}$$

near the crack tip. Of course, such a field leaves the crack faces traction free. The resulting field

$$\phi_m^f = \phi_m^s + \phi_m^r, \quad \rho_m^f = \rho_m^s + \rho_m^r \tag{24}$$

is referred to as a fundamental field of order m for the crack tip. The fundamental field has no body forces. It exists for all values of m except $m = 2$ where in this case we must insist on

$$\text{Im } A_2 = 0 \tag{25}$$

in order to ascertain the existence of the complementary regular field for $m = 2$. Altogether the fundamental field has no body forces and shows zero tractions on C_+, C_- and ∂R. Using (3), we find that the displacements of the fundamental field have the asymptotic form

$$w_m^f \sim A_m z^{-m/2} \quad \text{as } z \to 0 \quad \text{on } C_+, C_-. \tag{26}$$

3.2.2. *Elastic reciprocity*. We shall now analyze at $z = 0$ the regular field characterized by (20), (21a), (21b). The functions ϕ, ρ of that field will be referred to as ϕ_1, ρ_1 respectively. Let ϕ_2, ρ_2 describe a fundamental field of order m, as given by (24).

Consider an oriented closed path Ω (Fig. 2), which consists of ∂R, of crack segments C'_+, C'_- and of a circle Γ with radius $\gamma < L$ and also sufficiently small such that the interior of the path is a sub-domain of R. We shall refer to the union of C_+, C_- collectively as C; and similarly, C' denotes the crack segments C'_+, C'_-. In this sub-domain fields 1 and 2 are regular; the reciprocity theorem, applied to the two fields, yields $W^*(\Omega) = 0$ or, in more detail,

$$W^*(\Gamma) = -W^*(C') - W^*(\partial R). \tag{27}$$

It will be convenient to set

$$W^*(\Omega') = W_s^*(\Omega') + W_r^*(\Omega')$$

for certain paths Ω'; here the subscript s refers to the singular field ϕ_m^s, ρ_m^s and to field 1, while the subscript r pertains to the regular complementary field ϕ_m^r, ρ_m^r and to field 1.

Turning now to $W^*(\Gamma)$ we have $W^*(\Gamma) = W_s^*(\Gamma) + W_r^*(\Gamma)$; evidently $W_r^*(\Gamma) = O(\gamma)$. Due to (8) we find

$$W_s^*(\Gamma) = -\frac{1+\kappa}{2\mu} \text{Im} \int_\Gamma [\rho_m^s \, d\phi_1 + \phi_m^s \, d\rho_1] + \text{Im } G_s \Big]_{z'=\gamma e^{i\pi}}^{z'=\gamma e^{-i\pi}}; \tag{28}$$

but the associated G_s vanishes at the ends of Γ since $P_2 = 0$. The expressions (21a) and (22) permit us to write

$$W_s^*(\Gamma) = -\text{Im} \sum_{n=1}^{L} D_{mn} J_{n-m} \quad \text{with} \tag{29a}$$

$$D_{mn} = \frac{n}{2}[(-1)^{m+1}\bar{A}_m a_n + A_m b_n] \quad \text{and} \tag{29b}$$

$$J_k = \int_\Gamma z^{(k-2)/2} \, dz. \tag{29c}$$

We observe that $J_0 = -2\pi i$ and that

$$J_k = 0 \quad \text{for } k \neq 0 \text{ and even}. \tag{30}$$

For odd k we find

$$J_k = -\frac{4i}{k}\gamma^{k/2} \sin\left[\frac{k\pi}{2}\right]. \tag{31}$$

Using (21b) we may rewrite D_{mn}. We obtain in particular

$$D_{mn} = \frac{n}{2}[-\bar{A}_m a_n + A_m \bar{a}_n] \quad \text{for } m = \text{even}, n = \text{odd} \tag{32}$$

$$D_{mn} = -\frac{n}{2}[-\bar{A}_m a_n + A_m \bar{a}_n] + A_m f_{(n-2)/2} \quad \text{for } m = \text{odd}, n = \text{even}. \tag{33}$$

Also

$$D_{mn} = -\frac{m}{2}[\bar{A}_m a_m + A_m \bar{a}_m] + A_m f_{(m-2)/2} \quad \text{for } m = \text{even} \tag{34}$$

$$D_{mm} = \frac{m}{2}[\bar{A}_m a_m + A_m \bar{a}_m] \quad \text{for } m = \text{odd}. \tag{35}$$

From here on we pursue the cases of even and odd m separately.

3.2.3. *Fundamental fields of even order* (m = *even*). Using eqns (29–35) we find

$$W_s^*(\Gamma) = -2\pi m \text{ Re } [\bar{A}_m a_m] + 2\pi \text{ Re } [A_m f_{(m-2)/2}]. \tag{36}$$

Turning to the contributions of the crack segment C' we observe first that

$$W_s^*(C') = 0. \tag{37}$$

Indeed the singular field of ϕ_m^s, ρ_m^s has no tractions on the crack faces. Its displacements are the same on opposite crack points (a consequence of m = even). Since the tractions of field 1 are of the induced type their total work through the displacements of the singular field vanishes. We may now write

$$W^*(C') = W_r^*(C'). \tag{38}$$

Combining all partial results, we obtain from (27)

$$-2\pi m \text{ Re } [\bar{A}_m a_m] + 2\pi \text{ Re } [A_m f_{(m-2)/2}] + O(\gamma) = -W_r^*(C') - W^*(\partial R). \tag{39}$$

Next we let $\gamma \to 0$ and arrive at the relation

$$\text{Re } [\bar{A}_m a_m] = \frac{1}{2\pi m}[W_r^*(C) + W^*(\partial R)] + \text{Re}\left[A_m \frac{1}{m} f_{(m-2)/2}\right]. \tag{40}$$

Finally we express the W^*-terms on the right by the work integrals involved and arrive at the final form of the integration formula

$$\text{Re } [\bar{A}_m a_m] = \frac{1}{2\pi m}\left[\int_C [X_1 u_m^r + Y_1 v_m^r] \, ds + \int_{\partial R} [X_1 u_m^f + Y_1 v_m^f] \, ds\right] + \text{Re}\left[A_m \frac{1}{m} f_{(m-2)/2}\right]. \tag{41}$$

Here X_1, Y_1 represent the tractions of field 1; u_m^f, v_m^f are displacements of the fundamental field and u_m^r, v_m^r are those of the complementary regular field. We note that once u_m^f, v_m^f are known, the displacements u_m^r, v_m^r can be determined from

$$u_m^r = u_m^f - u_m^s, \quad v_m^r = v_m^f - v_m^s$$

where u_m^s, v_m^s are the displacements of the singular field given in (22).

Generally, we need two fundamental fields of even order m to determine the complex coefficient a_m. To find Re a_m, we can choose a fundamental field corresponding to $A_m = 1$ while we can take $A_m = \mathrm{i}$ for determining Im a_m. For $m = 2$, the restriction (25) implies that the coefficient Im a_2 cannot be determined by the integration formula and is left arbitrary. Fortunately, Im a_2 is only related to rigid body rotation and we can set it to zero for definiteness. The b_m follow from (21b).

3.2.4. *Fundamental fields of odd order* (m = *odd*). In this case, the analogue of (36) is

$$W_\gamma^*(\Gamma) = 2\pi m \text{ Re } [\bar{A}_m a_m] + H \quad \text{with} \tag{42a}$$

$$H = \sum_{n=\text{even}} \frac{4}{n-m} \gamma^{(n-m)/2} \sin [(n-m)\pi/2] \text{ Re } [A_m f_{(n-2)/2}]. \tag{42b}$$

Setting $m = 2k+1$ and $n = 2t$, we may also write

$$H = \sum_{t=1}^{\infty} 4\sqrt{\gamma} \frac{(-\gamma)^{t-k-1}}{2t-2k-1} \text{ Re } [A_{2k+1} f_{t-1}].$$

For $m > 1$ ($k > 0$), H is generally unbounded in the neighborhood of $\gamma = 0$. We therefore cannot expect $W_\gamma^*(\Gamma)$ to reach a limit as $\gamma \to 0$. But let us assume that

$$f_t = 0 \quad \text{for} \quad t < (m-1)/2 = k. \tag{43}$$

In this case we find $H = O(\sqrt{\gamma})$. Hence

$$W_\gamma^*(\Gamma) \to 2\pi m \text{ Re } [\bar{A}_m a_m] \quad \text{as } \gamma \to 0 \tag{44a}$$

and also

$$W^*(\Gamma) \to 2\pi m \text{ Re } [\bar{A}_m a_m] \quad \text{as } \gamma \to 0. \tag{44b}$$

Because of (43) the integral $W^*(C)$ exists and we have

$$\lim_{\gamma \to 0} W^*(C') = W^*(C). \tag{45}$$

All of these permit to state as analogue of (41)

$$\text{Re } [\bar{A}_m a_m] = -\frac{1}{2\pi m} \int_{C \cup \partial R} [X_1 u_m^f + Y_1 v_m^f] \, \mathrm{d}s. \tag{46}$$

For $m = 1$ this formula was first given by Bueckner (1970, 1973). In this special case the restriction (43) is void. The complex coefficient a_1 is related to the stress intensity factors K_I and K_II by

$$K_\mathrm{I} - \mathrm{i} K_\mathrm{II} = \sqrt{2\pi} a_1.$$

As before the coefficient a_m can be determined with the aid of two fundamental fields, characterized by $A_m = 1$ and $A_m = \mathrm{i}$. If the restriction (43) does not apply one can modify the field functions ϕ_1, ρ_1 into

$$\tilde{\phi}_1 = \phi_1, \quad \tilde{\rho}_1 = \rho_1 - p(z) \tag{47}$$

where $p(z)$ is the polynomial

$$p(z) = \sum_{k=1}^{n} \frac{1}{k} f_{k-1} z^k \quad \text{where } 2n+1 \geqslant m. \tag{48}$$

The regular field of $\tilde{\phi}_1, \tilde{\rho}_1$ abides by (43), and formula (46) becomes applicable with respect to the regular field $\tilde{\phi}_1, \tilde{\rho}_1$.

If there exists a small neighborhood, $-\varepsilon \leqslant x \leqslant 0$ on the crack faces near the crack tip which is free of tractions, the restriction (43) on or the modification (47) to the field ϕ_1, ρ_1 can be disregarded and the integration formula (46) is valid for any odd m. In addition, the integration formula (41) for even m can be simplified since all f_k vanish. Furthermore, as has already been explained, the work done by the tractions of field 1 through the displacements of ϕ^{s}_m, ρ^{s}_m, m = even, on the crack segment C is zero, and we can superpose the displacements u^{s}_m, v^{s}_m onto u^{r}_m, v^{r}_m in (41). Thus, when a traction-free neighborhood on the crack faces near the crack tip is present, the integration formulas (41) and (46) can be combined as

$$\mathrm{Re}\,[\bar{A}_m a_m] = \frac{(-1)^m}{2\pi m} \int_{C \cup \partial R} [X_1 u^{f}_m + Y_1 v^{f}_m]\, \mathrm{d}s; \quad \text{for all } m. \tag{49}$$

4. PROBLEMS OF THE THIRD KIND

When the plane body is subjected to a combination of continuously prescribed tractions and imposed displacements on the boundary, the integration formulas developed in the previous sections will have to be modified somewhat. In addition, new fundamental fields have to be introduced in order to determine the coefficients of series expansions which are related to rigid body translations.

4.1. *Interior points*

Let the body R with boundary ∂R of Fig. 1 be loaded by prescribed tractions on ∂R_T and under imposed displacements on ∂R_u, where $\partial R_T \cup \partial R_u = \partial R$. Because of the geometric boundary conditions on ∂R_u, we have to admit $a_0 \neq 0$, $b_0 \neq 0$ in the expansions of ϕ, ρ about $z = 0$ given in (10). The singular field ϕ^{s}_m, ρ^{s}_m of (11) is still essential but it will give rise to non-zero tractions on ∂R_T and displacements on ∂R_u. These surface tractions and displacements will be relieved by a complementary regular field, ϕ^{r}_m and ρ^{r}_m, in order to construct (12). It is clear that ϕ^{r}_m, ρ^{r}_m exist for *all* values of $m \geqslant 1$, because of the geometric boundary conditions on ∂R_u. Hence the restriction (13) for the case of $m = 1$ can be dropped. The fundamental field obtained by such a construction has no body forces and it induces zero tractions and displacements on ∂R_T and ∂R_u, respectively. Equations (14), (16) and (17) will still hold but (15) now becomes

$$W^*(\partial R) = W^*(\partial R_T \cup \partial R_u) = \mathrm{Im}\left[\int_{\partial R_T} \bar{w}_2\, \mathrm{d}P_1 - \int_{\partial R_u} \bar{w}_1\, \mathrm{d}P_2\right]; \tag{50}$$

formula (17) does not change if the summation is extended to $n = 0$, and the integration formula (19) is modified to

$$\mathrm{Re}\,[B_m a_m + A_m b_m] = -\frac{1}{2\pi m}\left[\int_{\partial R_T} [X_1 u_m^f + Y_1 v_m^f]\,\mathrm{d}s - \int_{\partial R_u} [X_m^f u_1 + Y_m^f v_1]\,\mathrm{d}s\right]; \quad \text{for } m > 0 \tag{51}$$

where X_m^f, Y_m^f are the tractions of the fundamental field of order m. So much for the coefficients a_m, b_m of subscript $m \geqslant 1$.

Turning now to the case of a_0, b_0 we observe that the rigid body translation at the origin is given by

$$2\mu w_1(0) = \kappa a_0 - \bar{b}_0. \tag{52}$$

This shows that the coefficients a_0, b_0 are partially redundant. Thus we can only expect to determine the preceding combination but not the individual coefficient. In order to develop an integration formula for $w_1(0)$ we consider the following singular field:

$$\phi_0^s = \frac{2\mu}{1+\kappa} A_0 \log z, \quad \rho_0^s = -\frac{2\mu}{1+\kappa}\kappa \bar{A}_0 \log z; \quad \log z = \log r + \mathrm{i}\theta. \tag{53}$$

This singular field yields single-valued displacements

$$w = \frac{2\kappa A_0}{1+\kappa}\log r + \frac{\bar{A}_0}{1+\kappa}(1 - \mathrm{e}^{2\mathrm{i}\theta}), \quad r \neq 0. \tag{54}$$

Furthermore

$$P_0^s = \frac{2\mu}{1+\kappa}[(1-\kappa)A_0 \log r + (1+\kappa)A_0 \mathrm{i}\theta - \bar{A}_0(1-\mathrm{e}^{2\mathrm{i}\theta})]. \tag{55}$$

The singular field ϕ_0^s, ρ_0^s gives non-zero tractions and displacements on ∂R_T and ∂R_u, respectively. We shall relieve them by a complementary regular field ϕ_0^r, ρ_0^r and construct a fundamental field of order zero as

$$\phi_0^f = \phi_0^s + \phi_0^r, \quad \rho_0^f = \rho_0^s + \rho_0^r. \tag{56}$$

As before, this fundamental field has no body forces and it induces zero tractions and displacements on ∂R_T and ∂R_u, respectively.

We now apply the reciprocity theorem to the regular field of ϕ_1, ρ_1 and the fundamental field of the pair ϕ_0^f, ρ_0^f. As before $W^*(\Gamma)$ and the limit procedure $\gamma \to 0$ are of central importance. In analogy to (16) we find

$$W^*(\Gamma) = -\frac{1+\kappa}{2\mu}\,\mathrm{Im}\int_{\zeta'}^{\zeta''} [\rho_0^s\,\mathrm{d}\phi_1 + \phi_0^s\,\mathrm{d}\rho_1] + \mathrm{Im}\,G_s\Big]_{\zeta'}^{\zeta''} + O(\gamma) \tag{57}$$

where $\zeta' = \gamma\mathrm{e}^{\mathrm{i}\pi}$ and $\zeta'' = \gamma\mathrm{e}^{-\mathrm{i}\pi}$. Since $\mathrm{d}\phi_1$, $\mathrm{d}\rho_1$ are independent of a_0, b_0 the integral in (57) will not contribute to a formula for $w_1(0)$; a crude estimate of the integral is

$$\int_{\zeta'}^{\zeta''} [\rho_0^s\,\mathrm{d}\phi_1 + \phi_0^s\,\mathrm{d}\rho_1] = O(\gamma|\log\gamma|).\dagger \tag{58}$$

As for the G_s-term in (57) we can observe that [see also (55)]

† One can show that the right hand side can be replaced by $O(\gamma)$.

$$G_s\Big]_{\zeta^-}^{\zeta^-} = -\bar{w}_1(-\gamma)P_0^s\Big]_{\zeta^-}^{\zeta^-} = 4\pi i\mu A_0\bar{w}_1(-\gamma) = 4\pi i\mu A_0\bar{w}_1(0)+O(\gamma), \tag{59}$$

so that

$$W^*(\Gamma) \to 4\pi\mu \text{ Re } [\bar{A}_0 w_1(0)] \quad \text{as } \gamma \to 0. \tag{60}$$

Combining this with (14) and (50) we obtain the integration formula

$$\text{Re } [\bar{A}_0 w_1(0)] = -\frac{1}{4\pi\mu}\left[\int_{\partial R_T}[X_1u_m^f + Y_1v_m^f]\,\mathrm{d}s - \int_{\partial R_u}[u_1X_m^f + v_1Y_m^f]\,\mathrm{d}s\right]. \tag{61}$$

The real and imaginary parts of $w_1(0)$ can be obtained by choosing A_0 to be 1 and i, respectively.

4.2. *Crack tips*

As in Section 4.1 we assume tractions on ∂R_T and displacement conditions on ∂R_u; the latter shall not involve the crack faces. We also assume (21a), (21b). The geometric conditions on the regular field ϕ_1, ρ_1 require the inclusion of $n = 0$ in the local expansions of the two analytic functions in (21a). But unlike the other complex coefficients in (21b), a_0, b_0 are unrelated. For $m \geqslant 1$ we shall still employ the singular field ϕ_m^s, ρ_m^s given in (22) for the cracked body but the complementary regular field ϕ_m^r, ρ_m^r of (23) is chosen such that the fundamental field ϕ_m^f, ρ_m^f constructed in (24) shows zero tractions on ∂R_T and zero displacements on ∂R_u. Further, the complementary regular field will exist for $m = 2$ even without the restriction (25) because of the presence of geometric boundary conditions on ∂R_u.

Following the basic procedure that led us to the integration formula (41) for the complex coefficient a_m, with even m, we obtain the modification

$$\text{Re } [\bar{A}_m a_m] = \frac{1}{2\pi m}\left[\int_C [X_1u_m^r + Y_1v_m^r]\,\mathrm{d}s + \int_{\partial R_T}[X_1u_m^f + Y_1v_m^f]\,\mathrm{d}s\right.$$
$$\left. - \int_{\partial R_u}[X_m^f u_1 + Y_m^f v_1]\,\mathrm{d}s\right] + \text{Re}\left[A_m\frac{1}{m}f_{(m-2)/2}\right]; \quad m \text{ even}. \tag{62}$$

For odd m, the integration formula (46) changes into

$$\text{Re } [\bar{A}_m a_m] = -\frac{1}{2\pi m}\left[\int_{C\cup\partial R_T}[X_1u_m^f + Y_1v_m^f]\,\mathrm{d}s - \int_{\partial R_u}[X_m^f u_1 + Y_m^f v_1]\,\mathrm{d}s\right]. \tag{63}$$

Further, when there is some neighborhood of the crack tip in which the crack faces are free of tractions, the presence of prescribed boundary displacements changes formula (49) to

$$\text{Re } [\bar{A}_m a_m] = \frac{(-1)^m}{2\pi m}\left[\int_{C\cup\partial R_T}[X_1u_m^f + Y_1v_m^f]\,\mathrm{d}s - \int_{\partial R_u}[X_m^f u_1 + Y_m^f v_1]\,\mathrm{d}s\right]; \quad \text{for all } m > 0. \tag{64}$$

In (62–64), X_m^f and Y_m^f are the tractions of the fundamental field ϕ_m^f, ρ_m^f.

As to the determination of $w_1(0)$, we consider the following singular field:

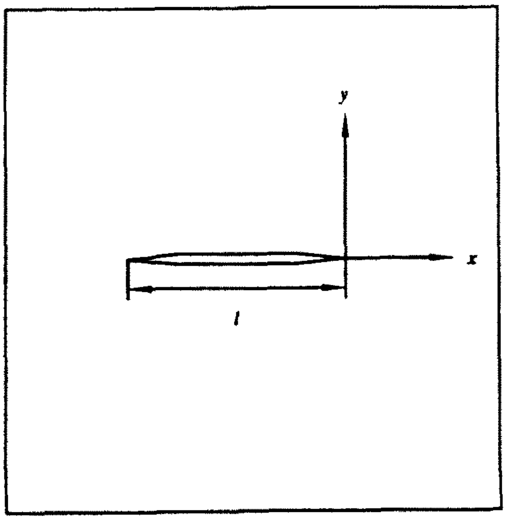

Fig. 3. A closed crack extending from $z = -l$ to $z = 0$ in a finite body R.

$$\phi_0^s = \mu A_0 \log z, \quad \rho_0^s = -\mu \bar{A}_0 \log z. \tag{65}$$

Such a field yields

$$P_0^s = \mu[2A_0 i\theta - (1 - e^{2i\theta})\bar{A}_0] \tag{66}$$

which is constant on $\theta = \pm\pi$ and by (2a) the tractions of ϕ_0^s, ρ_0^s vanish on the crack faces. The singular field ϕ_0^s, ρ_0^s is augmented in the usual manner by a complementary regular field ϕ_0^r, ρ_0^r to arrive at a fundamental field ϕ_0^f, ρ_0^f which has no body forces and gives zero tractions on C, ∂R_f and zero displacements on ∂R_u.

Following the procedure established in previous sections, we apply the reciprocity theorem to the two fields ϕ_1, ρ_1 and ϕ_0^f, ρ_0^f in the sub-domain of R bounded by the contour which consists of a circle Γ, of crack segments C'' and of ∂R_f and ∂R_u. We find

$$W_r^*(\Gamma) = O(\sqrt{\gamma}), \tag{67a}$$

$$W_s^*(\Gamma) = -\frac{1+\kappa}{2\mu}\,\mathrm{Im}\int_{C''}^{C''}[\rho_0^s\,\mathrm{d}\phi_1 + \phi_0^s\,\mathrm{d}\rho_1] + \mathrm{Im}\,G_s\Big]_{C''}^{C''}. \tag{67b}$$

Here the order relation (58) applies to the integral-term. As before the boundary term is as in (59) with $O(\gamma)$ replaced by $O(\sqrt{\gamma})$. Altogether, in the limit $\gamma \to 0$ the reciprocity theorem leads to an integration formula for the rigid body translation of the regular field at $z = 0$:

$$\mathrm{Re}\,[\bar{A}_0 w_1(0)] = -\frac{1}{4\pi\mu}\left[\int_{C \cup \partial R_f}[X_1 u_m^f + Y_1 v_m^f]\,\mathrm{d}s - \int_{\partial R_u}[X_m^f u_1 + Y_m^f v_1]\,\mathrm{d}s\right] \tag{68}$$

where Re $[w_1(0)]$ can be determined by choosing $A_0 = 1$ and Im $[w_1(0)]$ by setting $A_0 = \mathrm{i}$.

5. FUNDAMENTAL FIELDS FOR A CLOSED CRACK

The crack configuration which we have considered thus far is the open crack. It is clear that we cannot use the singular field of an open crack in a closed crack geometry because it would cause undue crack openings on some portions of the crack plane. However, guided by the procedure that we have developed for the open crack, we can construct the fundamental field for a closed crack in the following manner.

Take a finite body R which contains a crack, extending from $x = -l$ to $x = 0$, Fig. 3. The singular field ϕ_m^s, ρ_m^s given in (22) for m = even, $m \neq 0$, is still valid for this closed crack geometry since (22) yields continuous displacements in R outside the closed crack. But this is not so for odd m. We therefore shall construct an appropriate singular field of odd order for the closed crack geometry depicted in Fig. 3 as follows.

First, we define a polynomial of degree n in z, $q_n(z)$, as

$$q_n(z) \equiv \frac{1}{\sqrt{l}} \sum_{k=0}^{n} \binom{-\frac{1}{2}}{k} \left[\frac{z}{l}\right]^k; \quad n = 1,2,3,\ldots \tag{69a}$$

and let

$$F_0(z) \equiv \sqrt{(z+l)z}, \quad F_0 > 0 \quad \text{for real } z > 0. \tag{69b}$$

Then the function $F_n(z)$ defined by

$$F_n(z) \equiv q_{n-1}(z) F_0(z) z^{-n} \tag{70}$$

has branch points at $z = 0$ and $z = -l$ and is holomorphic in the z-plane outside the crack. In order to investigate the behavior of $F_n(z)$ near $z = 0$, we first note that (69a) can be rewritten as

$$q_n(z) = \frac{1}{\sqrt{l}} \left[1 + \frac{z}{l}\right]^{-1/2} - \frac{1}{\sqrt{l}} \sum_{k=n+1}^{\infty} \binom{-\frac{1}{2}}{k} \left[\frac{z}{l}\right]^k. \tag{71}$$

Then we have

$$q_{n-1}(z) F_0(z) = \sqrt{z}\left[1 - \sqrt{1+\frac{z}{l}} \sum_{k=n}^{\infty} \binom{-\frac{1}{2}}{k} \left[\frac{z}{l}\right]^k\right], \tag{72}$$

and since

$$\sqrt{1+\frac{z}{l}} = \sum_{j=0}^{\infty} \binom{\frac{1}{2}}{j} \left[\frac{z}{l}\right]^j,$$

we can rewrite (72) as

$$q_{n-1}(z) F_0(z) = \sqrt{z}\left[1 - \sum_{k=n}^{\infty} \gamma_k^{n-1} \left[\frac{z}{l}\right]^k\right] \tag{73}$$

with certain coefficients γ_k^{n-1}. Using (73) we can represent the function $F_n(z)$ near $z = 0$ in the form

$$F_n(z) = z^{-n+1/2} - l^{-n+1/2} \sqrt{z/l} \sum_{k=0}^{\infty} \gamma_{k+n}^{n-1} \left[\frac{z}{l}\right]^k. \tag{74}$$

Near $z = -l$ the same function admits an expansion of the form

$$F_n(z) = \sqrt{z+l} \sum_{k=0}^{\infty} \beta_k^n (z+l)^k. \tag{75}$$

Now with the help of $F_n(z)$ given in (70), we construct the following singular field of order $m = 2n-1$ for the closed crack geometry:

$$\phi_m^s = \frac{2\mu}{1+\kappa} A_m F_{(m+1)/2}(z), \quad \rho_m^s = \frac{2\mu}{1+\kappa} \bar{A}_m F_{(m+1)/2}(z); \quad m \text{ odd}. \tag{76}$$

This field is regular outside any neighborhood of $z = 0$. Within such a neighborhood it differs from (22) by a regular field. It can be shown that the field of (76) has no tractions on the crack faces. On any closed loop around the crack the tractions are self-equilibrated while the stresses vanish at infinity. Boundary tractions on ∂R_T and displacements on ∂R_u can be compensated to zero by a regular field ϕ_m^r, ρ_m^r. We therefore complement the singular field (76) by this regular field to construct, similar to (24), a fundamental field ϕ_m^f, ρ_m^f. As in the previous considerations, this fundamental field has no body forces and it shows zero tractions and displacements on ∂R_T and ∂R_u, respectively. With some minor modifications, the procedure given in the previous sections for applying the reciprocity theorem can be repeated for the closed crack geometry. We find that the complex coefficients a_m (m = odd) of the expansion of the elastic field about $z = 0$ can also be determined by the same formulas (46) [with appropriate restrictions (43), (47) and (48)] and (63).

To determine the rigid body translation at $z = 0$ in the case of problems of the third kind, we can employ the following singular field:

$$\phi_0^s = \frac{2\mu A_0}{1+\kappa}\left[\log z + \frac{\kappa-1}{2} h(z)\right], \quad \rho_0^s = \frac{2\mu \bar{A}_0}{1+\kappa}\left[-\kappa \log z + \frac{\kappa-1}{2} h(z)\right] \tag{77a}$$

where

$$h(z) = -2 \log\left[\frac{1+\sqrt{1+(l/z)}}{2}\right] = 2 \log\left[\frac{2\sqrt{z/(l+z)}}{1+\sqrt{z/(l+z)}}\right]. \tag{77b}$$

Here the square-roots are to be taken as the main branch, i.e. with values in the right-hand half-plane. This implies that the arguments under the log-function in (77b) are also in that half-plane. The main branch of log is to be used. The function $h(z)$ so defined is holomorphic outside the crack, $z = \infty$ included. On the crack it has the important property

$$\text{Re}\,[h(x)] = 2 \log 2 - \log l + \log |x|. \tag{78}$$

It follows from (55) and (78) that the singular field of (77a) has no tractions on the crack faces. The field is the response to the point force $Q = -4\pi\mu A_0$, attacking the end point $z = 0$ of the crack. At the other end point $z = -l$ the field behaves like a regular one.

With the aid of the singular field (77a, b) and its complementary regular field, we can construct an appropriate fundamental field for the closed crack geometry. Once again we can repeat, with some minor changes, the procedure of the reciprocity theorem and obtain the same integration formula as given in (68) for determining the rigid body translation at $z = 0$ in this closed crack geometry.

6. UNBOUNDED DOMAIN AND OTHER GENERALIZATIONS

So far the integration formulas for the coefficients a_m, b_m have been established for finite bodies. With suitable conditions on the surface tractions and geometric boundary

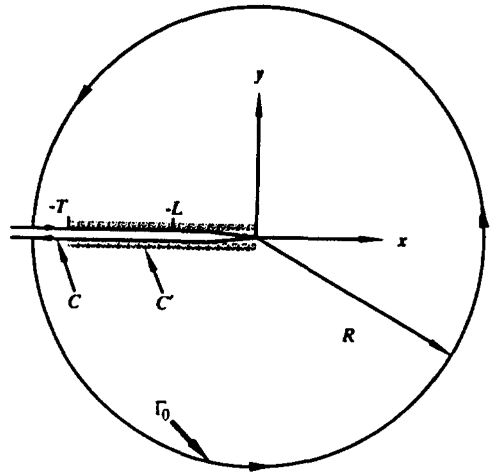

Fig. 4. A semi-infinite crack. Crack face tractions of the induced type are applied in the interval $(-T,0)$. In $(-L,0)$, these applied tractions are represented by the power series of (20). C denotes the upper and lower crack faces and C' is the crack segments within $(-T,0)$.

data the formulas can be extended to certain infinite bodies as well. We shall show this for the whole z-plane cracked along the negative real axis, Fig. 4.

6.1. *Semi-infinite crack in infinite domain*

Let the crack be loaded by tractions of the induced type; no other load (including at $z=\infty$) is admitted. Going back to (2a) we can describe the load with the aid of $P_1(x) = \phi_1(x)+\overline{\rho_1(x)}$. For a bounded and Holder-continuous $P_1(x)$ Bueckner (1970) has given the responding regular field in the form

$$\phi_1(z) = -\frac{\sqrt{z}}{2\pi}\int_{-\infty}^{0}\frac{P_1(t)\,\mathrm{d}t}{\sqrt{|t|}(t-z)},\quad \rho_1(z) = -\frac{\sqrt{z}}{2\pi}\int_{-\infty}^{0}\frac{\overline{P_1(t)}\,\mathrm{d}t}{\sqrt{|t|}(t-z)}. \tag{79}$$

Let now $P_1(t) \equiv 0$ for $t < -T < 0$ and let also $(-T,0)$ contain the interval $(-L,0)$ along which the tractions are prescribed in the form (20). From (79) it then follows that the quantities w_1, Z_1 obey the asymptotic relations

$$w_1 = O(|z|^{-1/2}),\quad Z_1 = O(|z|^{-3/2}),\quad \text{as } z\to\infty. \tag{80}$$

We now consider the regular field within the disk $|z| \leqslant R\ (>T)$. Its circular boundary is Γ_0; the crack portion within $(-T,0)$ is denoted by C'. We shall apply the coefficient formulas of Section 3 to this finite configuration in a modified form. To this end we observe that the formulas stay valid if we use $\phi_m' \equiv 0$, $\rho_m' \equiv 0$ in (24) while replacing $W^*(\partial R)$ in the form (15) by the extended and original form (7a).

For even $m\ (=2n)$ we obtain

$$\mathrm{Re}\,[\bar{A}_m a_m] = \mathrm{Re}\left[\bar{A}_m\frac{1}{m}\bar{f}_{n-1}\right]+K_0 \tag{81a}$$

with

$$K_0 = \mathrm{Re}\left[\frac{1}{2\pi m}\int_{\Gamma_0}[Z_1\bar{w}_m' - Z_m'\bar{w}_1]\,\mathrm{d}s\right]. \tag{81b}$$

From (22) and (80) it follows that $K_0 \to 0$ as $R\to\infty$, which in turn leads to

$$\mathrm{Re}\left[\bar{A}_m\left[a_m-\frac{1}{m}\bar{f}_{n-1}\right]\right]=0$$

for all A_m ($m \neq 2$) and all Re $[A_2]$ for $m = 2$. Disregarding the ambiguity of rigid body rotation we may write

$$a_m=\frac{1}{m}\bar{f}_{n-1}\quad \text{for all } m=2n\geqslant 2. \tag{82}$$

For odd m ($=2n+1$) we shall use (47), (48). Here we observe that

$$\tilde{w}_1=O(|z|^n),\quad \tilde{Z}_1=O(|z|^{n-1}),\quad \text{as } z\to\infty. \tag{83}$$

The analogue of (46) is

$$\mathrm{Re}\,[\bar{A}_m a_m]=-\frac{1}{2\pi m}\int_{C}[\tilde{X}_1 u_m'+\tilde{Y}_1 v_m']\,\mathrm{d}s+\tilde{K}_0 \tag{84a}$$

with

$$\tilde{K}_0=-\mathrm{Re}\left[\frac{1}{2\pi m}\int_{\Gamma_0}[\tilde{Z}_1\bar{w}_m'-Z_m'\bar{\tilde{w}}_1]\,\mathrm{d}s\right]. \tag{84b}$$

Now (22) and (83) yield $\tilde{K}_0\to 0$ as $R\to\infty$. Therefore

$$\mathrm{Re}\,[\bar{A}_m a_m]=-\frac{1}{2\pi m}\int_{C}[\tilde{X}_1 u_m'+\tilde{Y}_1 v_m']\,\mathrm{d}s. \tag{85}$$

Altogether we can now state that the functions (22) represent a fundamental field of order m for the z-plane with a crack along the negative real axis.

The integrand in (85) does not necessarily vanish for points $x < -T$ of the crack; due to (22), (83) it has order $O(|x|^{-3/2})$ as $x\to-\infty$. Since we deal with a load of the induced type the faces C_+ and C_- contribute equally to (85) and one may write

$$\mathrm{Re}\,[\bar{A}_m a_m]=-\frac{1}{\pi m}\int_{-\infty}^{0}[\tilde{X}_1 u_m'+\tilde{Y}_1 v_m']\,\mathrm{d}x,\quad \text{with integrand on } C_+. \tag{86}$$

We have to observe

$$\tilde{X}_1+\mathrm{i}\tilde{Y}_1=X_1+\mathrm{i}Y_1+\mathrm{i}\bar{p}'\quad \text{with}\quad p'(x)=\sum_{k=0}^{n-1}f_k x^k. \tag{87}$$

Along the segment $[-L,0]$ of $[-T,0]$ we have in particular

$$\tilde{X}_1+\mathrm{i}\tilde{Y}_1=-\mathrm{i}\sum_{k=n}^{\infty}\bar{f}_k x^k. \tag{88a}$$

and for $[-\infty,-T]$ we find

$$\tilde{X}_1+\mathrm{i}\tilde{Y}_1=\mathrm{i}\overline{p'(x)}=\mathrm{i}\sum_{k=0}^{n-1}\bar{f}_k x^k. \tag{88b}$$

Since P_m' vanishes along the crack the displacement w_m' takes the simple form

$$w_m^s = A_m z^{-m/2} \quad \text{on } C. \tag{89}$$

As a particular consequence (88b) and (89) we have

$$\int_{-\infty}^{-T} [\tilde{X}_1 u_m^s + \tilde{Y}_1 v_m^s]\,\mathrm{d}x = -\mathrm{Re}\left[A_m \int_{-\infty}^{-T} |x|^{-1/2} x^{-n} p'(x)\,\mathrm{d}x\right]$$

$$= \mathrm{Re}\left[A_m \sum_{k=0}^{n-1} (-1)^{k-n} \frac{f_k}{k-n+\frac{1}{2}} T^{k-n+1/2}\right]. \tag{90}$$

In a similar vein the integral over $(-L, 0)$ in (86) can be evaluated. One obtains

$$\int_{-L}^{0} [\tilde{X}_1 u_m^s + \tilde{Y}_1 v_m^s]\,\mathrm{d}x = \mathrm{Re}\left[A_m \sum_{k=n}^{\infty} (-1)^{k-n} \frac{f_k}{k-n+\frac{1}{2}} L^{k-n+1/2}\right]. \tag{91}$$

Since (86) holds for any A_m the coefficient a_m is found by using $A_m = 1$ and $A_m = \mathrm{i}$. We write here the final form for a_m in the case $T = L$:

$$a_m = -\frac{1}{\pi m} \sum_{k=0}^{\infty} (-1)^{k-n} \frac{\bar{f}_k}{k-n+\frac{1}{2}} L^{k-n+1/2}. \tag{92}$$

As before the coefficients b_m follow from (21b). Thus

$$b_m = \frac{1}{m} f_{n-1} \quad \text{for even } m = 2n \tag{93a}$$

$$b_m = \bar{a}_m \quad \text{for odd } m. \tag{93b}$$

6.2. *Applications*

Let the load be a constant pressure σ, confined to the interval $(-L, 0)$ of the crack. This means $f_0 = -\sigma$, $f_k = 0$ for $k \geqslant 1$ in (20). Formulas (82), (92), (93a, b) yield

$$a_2 = b_2 = \tfrac{1}{2} f_0 = -\tfrac{1}{2}\sigma; \quad a_{2n} = b_{2n} = 0 \quad \text{for } n > 1, \tag{94a}$$

$$a_{2n+1} = b_{2n+1} = \frac{-2(-1)^{-n}}{\pi(2n+1)(2n-1)} \sigma \sqrt{L} L^{-n} \quad \text{for } n = 0, 1, 2, \ldots \tag{94b}$$

With the aid of (79) one can determine ϕ_1, ρ_1 in closed form. Using

$$P_1(t) \equiv 0 \quad \text{for } t < -L; \quad P_1(t) = -\sigma(L+t) \quad \text{for } -L \leqslant t \leqslant 0, \tag{95}$$

one finds

$$\phi_1 = \rho_1 = \frac{\sqrt{z}}{2\pi} \int_{-L}^{0} \frac{\sigma(L+t)\,\mathrm{d}t}{\sqrt{|t|}(t-z)} = \frac{\sigma}{2\pi}\sqrt{z}(L+z) \int_{-L}^{0} \frac{\mathrm{d}t}{\sqrt{|t|}(t-z)} + \frac{\sigma}{2\pi}\sqrt{z} \int_{-L}^{0} \frac{\mathrm{d}t}{\sqrt{|t|}}. \tag{96}$$

Furthermore

$$\phi^*(z) \equiv \frac{\sqrt{z}}{2\pi} \int_{-L}^{0} \frac{\mathrm{d}t}{\sqrt{|t|}(t-z)} = -\frac{1}{2} + \frac{1}{2\pi \mathrm{i}} \log \frac{\sqrt{L} + \mathrm{i}\sqrt{z}}{\sqrt{L} - \mathrm{i}\sqrt{z}}, \tag{97}$$

so that

$$\phi_1 = \rho_1 = \sigma(L+z)\phi^*(z) + \frac{\sigma}{\pi}\sqrt{Lz}. \tag{98}$$

The function $\phi^*(z)$ admits the expansion

$$\phi^*(z) = -\frac{1}{2} + \frac{1}{\pi}(z/L)^{1/2} \sum_{k=0}^{\infty} \frac{(-1)^k (z/L)^k}{2k+1} \quad \text{for } |z| < L. \tag{99}$$

One can use (98) and (99) to confirm (94a, b).

From (97) and (98) one can derive the response to a point force pressure Q at $x = -L$. Elementary steps yield the new field functions

$$\phi_1 = \rho_1 = Q\frac{\partial}{\partial L}\left[(L+z)\phi^* + \frac{1}{\pi}\sqrt{Lz}\right] = Q\phi^*. \tag{100}$$

Although the formulas for the coefficients a_m, b_m have been derived with a view to regular fields with continuous distribution of prescribed boundary data, the formulas of the previous sections have more general validity. In particular (86) is applicable to the case on hand with $\tilde{X}_1 = X_1$, $\tilde{Y}_1 = Y_1$ and yields immediately

$$\begin{aligned} \mathrm{Re}\,[\bar{A}_m a_m] &= -\frac{Q}{\pi m} v_m^s \Big]_{x=-L} \\ &= \mathrm{Re}\,[A_m] \frac{Q}{\pi m}(-1)^n L^{-n-1/2}, \quad m = 2n+1. \end{aligned} \tag{101}$$

This agrees with (99), (100). The coefficients of even order, $a_2, a_4, \ldots$, and $b_2, b_4, \ldots$ vanish.

6.3. *Body forces*

If the regular elastic field, which we wish to analyze, is also caused by body forces then there are no analytic functions $\phi(z)$, $\rho(z)$ to describe the field, and it makes little sense to look for expansions (10) or (21a) as the case may be. There is a significant exception. If a neighborhood of the origin $z = 0$ is free from body forces, then ϕ, ρ exist in that neighborhood. The expansions (10) or (21a), the latter under condition (20), exist as before. All of the fundamental fields of this paper can be used in applications of the reciprocity theorem, and formulas for the coefficients a_m, b_m can be derived. They differ from (19), (41), (46) etc., in only one respect: wherever the work integral

$$\int_{\partial R} [X_1 u_m^f + Y_1 v_m^f]\, ds$$

appears, it has to be augmented by the analogous work of the body forces through the displacements of the fundamental field. The same applies to the modification (47).

If the body forces are everywhere, but of a simple nature, e.g. gravity, centrifugal forces, it is recommended to split the regular field F into two fields $F = F_1 + F_2$ where F_1 responds to the body forces regardless of boundary conditions in as simple a way as possible. Frequently F_1 can be found in closed and elementary form, for example Bueckner and Giaever (1966). To the analysis of F_2 all the formulas of this paper can be applied.

7. CONCLUDING REMARKS

In this paper, we have generalized Bueckner's fundamental field concept and developed a theory of higher order weight functions for computing expansion coefficients for both interior points and crack tips in linear elastic solids. For the expansion coefficients at crack tips, we have considered the fundamental fields for both open and closed crack geometries. By following the procedure of Bueckner (1970), our results could be extended to analytic cracks as well.

In general, closed form expressions of the fundamental fields and higher order weight functions are limited to simple geometries and numerical procedures are required to compute these higher order weight functions. A variational principle developd by Sham (1987) for determining singular fields in elastic bodies of finite size can be used to this purpose. A finite element implementation of the variational principle has been carried out for computing fundamental fields of first order (for calculating stress intensity factors) in two dimensions (Sham, 1987), and in three dimensions (Sham and Zhou, 1989), and an implementation has also been performed for computing fundamental fields for interface notch tip in anti-plane strain (Sham 1988a). In this procedure, only a single finite element analysis of the given geometry with fixed boundary (traction versus displacement) partition is required to generate the fundamental field. The same variational principle can also be used to obtain a finite element procedure for computing the higher order weight functions. It is reported in a separate work (Sham, 1988b).

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